

## RECURSIVE FUNCTIONS MODULO CO- $r$ -MAXIMAL SETS<sup>(1)</sup>

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**Abstract.** Define the equivalence relation  $\sim_A$  on the set of recursive functions of one variable by  $f \sim_A g$  if and only if  $f(x) = g(x)$  for all but finitely many  $x \in \bar{A}$ , where  $\bar{A}$  is an  $r$ -cohesive set, to obtain the structure  $\mathcal{R}/\bar{A}$ . Then the recursive functions modulo such an equivalence relation form a semiring with no zero divisors. It is shown that if  $A$  is  $r$ -maximal, then the structure obtained above is not a nonstandard model for arithmetic, a result due to Feferman, Scott, and Tennenbaum. Furthermore, if  $A$  and  $B$  are maximal sets, then a necessary and sufficient condition for  $\mathcal{R}/\bar{A}$  and  $\mathcal{R}/\bar{B}$  to be elementarily equivalent is obtained. It is also shown that many different elementary theories can be obtained for  $\mathcal{R}/\bar{A}$  by proper choice of  $\bar{A}$ .

**1. Introduction.** In 1934, Skolem [12, Theorem 4] proved the nonaxiomatizability of arithmetic by constructing nonstandard models for arithmetic. He did so by taking all arithmetic functions of one variable, and factoring out by the equivalence relation  $f \sim g$  if and only if  $f(x) = g(x)$  almost everywhere (a.e.) in  $A$ , i.e., for all but finitely many  $x \in A$ , where  $A$  is any arithmetically indecomposable set. The model thus obtained satisfies exactly the same first order sentences as the nonnegative integers. Tennenbaum raised the question as to whether, starting with the recursive functions of one variable, and obtaining a model by the above procedure, letting  $A$  be an  $r$ -cohesive set, would also produce nonstandard models for arithmetic. In 1959, Feferman, Scott, and Tennenbaum [1] answered this question in the negative. They showed, in fact, that any homomorphic image of the recursive functions which is not isomorphic to the semiring of integers is not even a model for the provable sentences of arithmetic.

For the sake of completeness, we now define some of the terminology and notation we will be using below. Other definitions will appear in the appropriate sections.

A *recursive function*  $f$ , intuitively, is a total function for which there exists an algorithm such that, for any  $x \in N$  (the nonnegative integers), the algorithm will

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Received by the editors April 29, 1968 and, in revised form, November 15, 1968.

*AMS Subject Classifications.* Primary 0257, 0270.

*Key Words and Phrases.* Nonstandard model for arithmetic,  $r$ -cohesive set,  $r$ -maximal set, retraceable set, elementary equivalence, many-one degree, Turing degree.

<sup>(1)</sup> This research was supported in part by National Science Foundation grant GP-7524, and contains part of the author's doctoral dissertation written under the direction of Professor Anil Nerode.

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give us a value for  $f(x)$  after a sufficiently finite amount of computation. A rigorous definition of recursive function can be found in [8, p. 121]. A set is *recursively enumerable* (r.e.) if and only if it is the range of a recursive function, or is empty. A set is *recursive* if and only if both it and its complement are r.e. A *partial recursive function* (p.r.f.)  $f$  is a function whose domain is an r.e. set, and for which there exists an algorithm as above to calculate  $f(x)$  for any  $x$  in the domain of  $f$ . The range of a p.r.f. is also an r.e. set. The notion of an r.e. set was first extensively studied by Post [9].

We deal, basically, with two types of r.e. sets, and their complements. For any set  $X$ , let  $\bar{X}$  denote the complement of  $X$ . A set  $C$  is *cohesive* (*r-cohesive*) if and only if  $C$  is infinite, and for any r.e. (recursive) set  $B$ , either  $C \cap B$  or  $C \cap \bar{B}$  is finite. A set  $X$  is *maximal* (*r-maximal*) if and only if it is r.e., and  $\bar{X}$  is cohesive (*r-cohesive*). Friedberg [2, Theorem 2] first proved the existence of maximal sets, and showed that every maximal set is hyperhypersimple. Robinson [11, Theorem 2] and Lachlan [4, Theorem 8] proved that there are *r-maximal* sets which are not hyperhypersimple, hence they cannot be maximal.

For any set  $A$ , let  $\mathcal{R}/A$  denote the structure obtained from the recursive functions of one variable under the identification  $f \sim g$  if and only if  $f(x) = g(x)$  a.e. in  $\bar{A}$ . In §2, we show that if  $A$  is *r-maximal*, then  $\mathcal{R}/\bar{A}$  has no zero divisors. The aim of this section is to find a necessary and sufficient condition for  $\mathcal{R}/\bar{A}$  and  $\mathcal{R}/\bar{B}$  to be elementarily equivalent ( $\equiv$ ) where  $A$  and  $B$  are maximal sets. We first reproduce the Feferman-Scott-Tennenbaum result mentioned above. Next we prove that if  $A$  and  $B$  are maximal sets, then  $\mathcal{R}/\bar{A} \equiv \mathcal{R}/\bar{B}$  if and only if  $\bar{A}$  and  $\bar{B}$  have the same many-one degree ( $\bar{A} =_m \bar{B}$ ). In this case, there is a recursive isomorphism between  $\mathcal{R}/\bar{A}$  and  $\mathcal{R}/\bar{B}$ . A theorem of the author [5, Theorem 1] shows that there are  $\aleph_0$  distinct many-one degrees of maximal sets contained in any Turing degree of a maximal set, so we conclude that the elementary theory of  $\mathcal{R}/\bar{A}$  for  $A$  a maximal set, varies within any given Turing degree of a maximal set.

**2. A necessary and sufficient condition for the elementary equivalence of the  $\mathcal{R}/\bar{A}$ .** The hope of getting nonstandard models for arithmetic from the structures  $\mathcal{R}/\bar{A}$  with  $\bar{A}$  *r-cohesive* arose by direct analogy with Skolem's construction [12].  $\mathcal{R}/\bar{A}$  is constructed in the same way that Skolem constructed his models, by replacing "arithmetic" with "recursive". The structure thus obtained is easily seen to be a semiring, with addition and multiplication defined pointwise. Furthermore, it has no zero divisors. For assume  $f \cdot g = 0$  a.e. in  $\bar{A}$ , i.e., for all but finitely many  $x \in \bar{A}$ ,  $f(x) \cdot g(x) = 0$ . Let  $B = \{x : f(x) = 0\}$ , and  $C = \{x : g(x) = 0\}$ . Clearly, both  $B$  and  $C$  are recursive. Since  $\bar{A}$  is infinite, we must have either  $B \cap \bar{A}$  is infinite, or  $C \cap \bar{A}$  is infinite, else  $f \cdot g \neq 0$ . By symmetry, assume  $B \cap \bar{A}$  is infinite. Since  $\bar{A}$  is *r-cohesive*, we must have  $\bar{A} \cap \bar{B} = \bar{A} - (B \cap \bar{A})$  finite, hence there exists a finite set  $F$  such that  $\bar{A} \subset B \cup F$ . Thus  $f(x) = 0$  a.e. on  $\bar{A}$ , i.e.  $f \sim 0$ , which is what was to be shown. However, Feferman, Scott, and Tennenbaum [1] proved a result which shows that if  $\bar{A}$  is infinite, then  $\mathcal{R}/\bar{A}$  is not a nonstandard model for arithmetic. Before

we proceed with the presentation of this result, we need several definitions and a lemma.

We assume a fixed language  $L$ , consisting of the binary relation symbol  $=$  (equality), the binary function symbols  $+$  (addition) and  $\cdot$  (multiplication), the binary relation symbol  $<$  (less than), individual constants  $0, 1, \dots$  for all the natural numbers, individual variable symbols  $x_1, x_2, \dots$ , and the logical connectives and quantifiers, parentheses, and comma. Our relational system is thus  $\langle N', +, \cdot, =, < \rangle = \mathfrak{B}$ . We assume also, the axioms defining equality, the axioms defining  $<$  to be a linear ordering, an axiom system for arithmetic (cf. [8, p. 103]), and the axioms

$$\mathbf{n} = 1 + \cdot \dots = 1 \quad \text{for all } n \in N.$$

( $n$  times)

Let  $\mathcal{N} = \langle N, +, \cdot, =, < \rangle$ , i.e., let  $\mathcal{N}$  be the standard model for arithmetic. Then clearly  $\mathcal{N}$  is an interpretation of  $\mathfrak{B}$  satisfying all the axioms. Let  $A$  be any  $r$ -maximal set. Then  $\mathfrak{A} = \langle \mathcal{R}/\bar{A}, +_A, \cdot_A, =_A, <_A \rangle$  is an interpretation of  $\mathfrak{B}$ , where if  $R_A \in \{=_A, +_A, \cdot_A, <_A\}$  and  $f_1^*, f_2^*, \dots \in \mathcal{R}/\bar{A}$ , then  $R_A(f_1^*, f_2^*, \dots)$  if and only if  $R(f_1(u), f_2(u), \dots)$  a.e. in  $\bar{A}$ . For any recursive function  $f$ , let  $f^*$  denote the equivalence class of  $f$  in  $\mathfrak{A}$ . Henceforth we will identify  $\mathcal{R}/\bar{A}$  with  $\mathfrak{A}$  and  $\mathcal{N}$  with  $N$ . This should cause no confusion.

An  $n$ -ary number theoretic relation  $R$  is said to be *expressible* if and only if there is a formula  $S$  of  $L$  with exactly  $n$  free variables such that for any  $k_1, \dots, k_n \in N$

- (1) if  $R(k_1, \dots, k_n)$  is true, then  $\vdash S(\mathbf{k}_1, \dots, \mathbf{k}_n)$ ; and
- (2) if  $R(k_1, \dots, k_n)$  is false, then  $\vdash \neg S(\mathbf{k}_1, \dots, \mathbf{k}_n)$ .

An  $n$ -ary number theoretic function  $f$  is said to be *representable* if and only if there is a formula  $S$  of  $L$  with exactly  $n+1$  free variables such that for any  $k_1, \dots, k_{n+1} \in N$

- (3) if  $f(k_1, \dots, k_n) = k_{n+1}$  then  $\vdash S(\mathbf{k}_1, \dots, \mathbf{k}_{n+1})$ ; and
- (4)  $\vdash (\exists! \mathbf{x}_{n+1}) S(\mathbf{k}_1, \dots, \mathbf{k}_n, \mathbf{x}_{n+1})$ .

We remark that every recursive relation is expressible and every recursive function is representable [8, pp. 131–134].

Let  $R$  be any  $n$ -ary number theoretic relation, and  $S$  a formula of  $L$  with exactly  $n$  free variables. We say that  $S$  *serves*  $R$  if and only if, for all  $k_1, \dots, k_n \in N$ ,

- (5)  $\mathcal{N} \models S(k_1, \dots, k_n)$  if and only if  $R(k_1, \dots, k_n)$  is true; and

(6) for any  $r$ -cohesive set  $\bar{A}$ , and recursive functions  $f_1, \dots, f_n$ , if  $f_1^*, \dots, f_n^*$  are the equivalence classes of  $f_1, \dots, f_n$  in  $\mathcal{R}/\bar{A}$ , then  $\mathcal{R}/\bar{A} \models S(f_1^*, \dots, f_n^*)$  if and only if  $R(f_1(u), \dots, f_n(u))$  is true a.e. in  $\bar{A}$ .

Let  $f$  be any  $n$ -ary number theoretic function, and  $S$  a formula of  $L$  with exactly  $n+1$  free variables. We say that  $S$  *serves*  $f$  if and only if, for all  $k_1, \dots, k_{n+1} \in N$ ,

- (7)  $\mathcal{N} \models S(k_1, \dots, k_{n+1})$  if and only if  $f(k_1, \dots, k_n) = k_{n+1}$ ; and

(8) for any  $r$ -cohesive set  $\bar{A}$ , and recursive functions  $f_1, \dots, f_{n+1}$ , if  $f_1^*, \dots, f_{n+1}^*$  are the equivalence classes of  $f_1, \dots, f_{n+1}$  respectively in  $\mathcal{R}/\bar{A}$ , then

$$\mathcal{R}/\bar{A} \models S(f_1^*, \dots, f_{n+1}^*)$$

if and only if  $f(f_1(u), \dots, f_n(u)) = f_{n+1}(u)$  a.e. in  $\bar{A}$ .

**LEMMA 2.1.** *Let  $R$  be any  $n$ -ary recursive relation. Then there is a formula  $S$  of  $L$  with exactly  $n$  free variables such that  $S$  serves  $R$ .*

**Proof.** We first show that if  $f$  is any  $n$ -ary recursive function, then there is a formula  $S$  of  $L$  which represents  $f$  and also serves  $f$ .

By the above remarks, we know that there is a formula  $S$  of  $L$  which represents  $f$ . Assume  $f(k_1, \dots, k_n) = k_{n+1}$ . By (3),  $\vdash S(\mathbf{k}_1, \dots, \mathbf{k}_{n+1})$ . Since  $\mathcal{N}$  is a model for arithmetic,  $\mathcal{N} \models S(k_1, \dots, k_{n+1})$ . Now assume  $\mathcal{N} \models S(k_1, \dots, k_{n+1})$ . Then since  $\mathcal{N}$  is a model for arithmetic,  $\neg \vdash \neg S(\mathbf{k}_1, \dots, \mathbf{k}_{n+1})$ . But if  $f(k_1, \dots, k_n) = u \neq k_{n+1}$ , then  $\vdash S(\mathbf{k}_1, \dots, \mathbf{k}_n, \mathbf{u})$  by (3), and by (4),  $\vdash S(\mathbf{k}_1, \dots, \mathbf{k}_n, \mathbf{x}_{n+1}) \supset \mathbf{x}_{n+1} = \mathbf{u}$ , i.e.,  $\vdash \mathbf{x}_{n+1} \neq \mathbf{u} \supset \neg S(\mathbf{k}_1, \dots, \mathbf{k}_n, \mathbf{x}_{n+1})$ . But  $\vdash \mathbf{k}_{n+1} \neq \mathbf{u}$  hence  $\vdash \neg S(\mathbf{k}_1, \dots, \mathbf{k}_{n+1})$ , contradiction. Hence we must have  $f(k_1, \dots, k_n) = k_{n+1}$ , and so (7) is satisfied.

To show that (8) is satisfied, we have six cases.

*Case 1.*  $f$  is the zero function;  $f(x) = 0$  for all  $x$ . Then by [8, p. 118],  $f$  is representable by  $S: \mathbf{x}_1 = ' \mathbf{x}_1 \ \& \ \mathbf{x}_2 = ' \mathbf{0}$ . Assume  $\mathcal{R}/\bar{A} \models S(f_1^*, f_2^*)$ . Then  $f_1^* =_A f_1^* \ \& \ f_2^* =_A 0^*$ . By definition of  $=_A$ , we have  $f_2(u) = 0$  for all but finitely many  $u \in \bar{A}$ . But  $f(f_1(u)) = 0$  for all  $u$ . Hence  $f(f_1(u)) = f_2(u)$  a.e. in  $\bar{A}$ . Conversely, assume  $f(f_1(u)) = f_2(u)$  a.e. in  $\bar{A}$ . Then since  $f(f_1(u)) = 0$  for all  $u$ ,  $f_2(u) = 0$  a.e. in  $\bar{A}$ , hence  $f_2^* =_A 0^*$ . Trivially,  $f_1^* =_A f_1^*$ , hence  $f_1^* =_A f_1^* \ \& \ f_2^* =_A 0^*$ . Thus  $\mathcal{R}/\bar{A} \models S(f_1^*, f_2^*)$ , and (8) is satisfied.

*Case 2.*  $f$  is the successor function;  $f(x) = x + 1$ . Then by [8, p. 118]  $f$  is representable by  $S: \mathbf{x}_2 = \mathbf{x}_1 + ' \mathbf{1}$ . Assume  $\mathcal{R}/\bar{A} \models S(f_1^*, f_2^*)$ . Then  $f_2^* =_A f_1^* +_A 1^*$ . Clearly  $f_1^* +_A 1^* =_A (f_1 + 1)^*$ . By definition of  $=_A$ , we have  $f_2(u) = (f_1 + 1)(u)$  a.e. in  $\bar{A}$ , i.e.,  $f_2(u) = f_1(u) + 1$  a.e. in  $\bar{A}$ . But  $f(f_1(u)) = f_1(u) + 1$  for all  $u$ . Hence  $f(f_1(u)) = f_2(u)$  a.e. in  $\bar{A}$ . Conversely, assume  $f(f_1(u)) = f_2(u)$  a.e. in  $\bar{A}$ . Then  $f_1(u) + 1 = f_2(u)$  a.e. in  $\bar{A}$ . Hence  $f_1^* +_A 1^* =_A (f_1 + 1)^* =_A f_2^*$ , so  $\mathcal{R}/\bar{A} \models S(f_1^*, f_2^*)$  and (8) is satisfied.

*Case 3.*  $f$  is the projection function;  $f(x_1, \dots, x_n) = x_i$ . Then by [8, p. 118]  $f$  is representable by  $S$ :

$$\mathbf{x}_1 = ' \mathbf{x}_1 \ \& \ \dots \ \& \ \mathbf{x}_n = ' \mathbf{x}_n \ \& \ \mathbf{x}_{n+1} = ' \mathbf{x}_i.$$

Assume  $\mathcal{R}/\bar{A} \models S(f_1^*, \dots, f_{n+1}^*)$ . Then  $f_{n+1}^* =_A f_i^*$ , hence  $f_{n+1}(u) = f_i(u)$  a.e. in  $\bar{A}$ . But  $f(f_1(u), \dots, f_n(u)) = f_i(u)$  for all  $u$ , hence  $f_{n+1}(u) = f(f_1(u), \dots, f_n(u))$  a.e. in  $\bar{A}$ . Conversely, assume  $f_{n+1}(u) = f(f_1(u), \dots, f_n(u))$  a.e. in  $\bar{A}$ . Then  $f_i(u) = f_{n+1}(u)$  a.e. in  $\bar{A}$ , hence  $f_i^* =_A f_{n+1}^*$ . Trivially,  $f_1^* =_A f_1^* \ \& \ \dots \ \& \ f_n^* =_A f_n^*$ , hence

$$\mathcal{R}/\bar{A} \models S(f_1^*, \dots, f_{n+1}^*)$$

and (8) is satisfied.

*Case 4.* Substitution. Assume  $g$ , an  $m$ -ary recursive function, and  $h_1, \dots, h_m$ ,  $n$ -ary recursive functions, are served and represented by formulas  $B, A_1, \dots, A_m$

respectively. Let  $f$  be the recursive function  $f(x_1, \dots, x_n) = g(h_1(x_1, \dots, x_n), \dots, h_m(x_1, \dots, x_n))$ . Then by [8, p. 118],  $f$  is represented by  $S$ :

$$(\exists y_1) \cdots (\exists y_m) (A_1(x_1, \dots, x_n, y_1) \& \cdots \& A_m(x_1, \dots, x_n, y_m) \& B(y_1, \dots, y_m, x_{n+1})).$$

Assume  $\mathcal{R}/\bar{A} \models S(f_1^*, \dots, f_{n+1}^*)$ . Then there are  $g_1^*, \dots, g_m^* \in \mathcal{R}/\bar{A}$  such that

$$\mathcal{R}/\bar{A} \models A_1(f_1^*, \dots, f_n^*, g_1^*) \& \cdots \& A_m(f_1^*, \dots, f_n^*, g_m^*) \& B(g_1^*, \dots, g_m^*, f_{n+1}^*).$$

Hence if  $1 \leq i \leq m$ ,  $\mathcal{R}/\bar{A} \models A_i(f_1^*, \dots, f_n^*, g_i^*)$  and  $\mathcal{R}/\bar{A} \models B(g_1^*, \dots, g_m^*, f_{n+1}^*)$ . By induction, we must have  $h_i(f_1(u), \dots, f_n(u)) = g_i(u)$  a.e. in  $\bar{A}$  for  $1 \leq i \leq m$ , and  $g(g_1(u), \dots, g_m(u)) = f_{n+1}(u)$  a.e. in  $\bar{A}$ . Hence for all but finitely many  $u \in \bar{A}$ ,

$$f(f_1(u), \dots, f_n(u)) = g(h_1(f_1(u), \dots, f_n(u)), \dots, h_m(f_1(u), \dots, f_n(u))) = f_{n+1}(u).$$

Conversely, assume  $f(f_1(u), \dots, f_n(u)) = f_{n+1}(u)$  a.e. in  $\bar{A}$ . By definition,

$$f(f_1(u), \dots, f_n(u)) = g(h_1(f_1(u), \dots, f_n(u)), \dots, h_m(f_1(u), \dots, f_n(u))).$$

Hence

$$f_{n+1}(u) = g(h_1(f_1(u), \dots, f_n(u)), \dots, h_m(f_1(u), \dots, f_n(u)))$$

a.e. in  $\bar{A}$ . For  $1 \leq i \leq m$ , define recursive functions  $g_i(u) = h_i(f_1(u), \dots, f_n(u))$ . Then  $g(g_1(u), \dots, g_m(u)) = f_{n+1}(u)$  a.e. in  $\bar{A}$ , so  $g(g_1^*, \dots, g_m^*) =_A f_{n+1}^*$  by definition of  $=_A$ . Also,  $h_i(f_1(u), \dots, f_n(u)) = g_i(u)$  for  $1 \leq i \leq m$ , so  $h_i(f_1^*, \dots, f_n^*) =_A g_i^*$ . By induction, for  $1 \leq i \leq m$ ,  $\mathcal{R}/\bar{A} \models A_i(f_1^*, \dots, f_n^*, g_i^*)$  and  $\mathcal{R}/\bar{A} \models B(g_1^*, \dots, g_m^*, f_{n+1}^*)$ . Hence  $\mathcal{R}/\bar{A} \models S(f_1^*, \dots, f_{n+1}^*)$  and (8) is satisfied.

*Case 5.* Recursion rule:  $f(x_1, \dots, x_n, 0) = g(x_1, \dots, x_n)$  and  $f(x_1, \dots, x_n, y+1) = h(x_1, \dots, x_n, y, f(x_1, \dots, x_n, y))$ , where  $g$  and  $h$  are served and representable by  $F(x_1, \dots, x_{n+1})$  and  $G(x_1, \dots, x_{n+3})$  respectively. Let  $\beta(x_1, x_2, x_3)$  be the Gödel  $\beta$  function;  $\beta(x_1, x_2, x_3)$  is the remainder obtained upon dividing  $1 + (x_3 + 1) \cdot x_2$  by  $x_1$ . By [8, p. 131]  $\beta$  is representable by

$$Bt(x_1, x_2, x_3, x_4) : (\exists w)(x_1 = ' (1 + '(x_3 + '1) \cdot 'x_2) \cdot 'w + 'x_4 \& x_4 < 1 + '(x_3 + '1) \cdot 'x_2).$$

We first show that  $\beta$  is served by  $Bt$ . Suppose  $\mathcal{R}/\bar{A} \models Bt(f_1^*, f_2^*, f_3^*, f_4^*)$ . Then there is a recursive function  $f_5$  such that

$$f_1^* =_A (1^* +_A (f_3^* +_A 1^*) \cdot_A f_2^*) \cdot_A f_5^* +_A f_4^* \& f_4^* <_A (1^* +_A (f_3^* +_A 1^*) \cdot_A f_2^*).$$

By the definition of  $=_A$ ,  $+_A$ ,  $\cdot_A$ ,  $<_A$ , we must have

$$f_1(u) = (1 + (f_3(u) + 1) \cdot f_2(u)) \cdot f_5(u) + f_4(u) \& f_4(u) < 1 + (f_3(u) + 1) \cdot f_2(u)$$

for all but finitely many  $u \in \bar{A}$ . Hence a.e. in  $\bar{A}$ ,

$$(\exists w)[(1 + (f_3(u) + 1) \cdot f_2(u)) \cdot w + f_4(u) = f_1(u) \& f_4(u) < 1 + (f_3(u) + 1) \cdot f_2(u)],$$

hence  $\beta(f_1(u), f_2(u), f_3(u)) = f_4(u)$  for all but finitely many  $u \in \bar{A}$ .

Conversely, assume  $\beta(f_1(u), f_2(u), f_3(u)) = f_4(u)$  a.e. in  $\bar{A}$ . For all  $u$ , since  $\beta$  is total, let  $f_4'(u) = \beta(f_1(u), f_2(u), f_3(u))$ . Then  $f_4^* = f_4'^*$ . Since  $Bt$  represents  $\beta$ , for each

$u$  there is a  $v$  such that  $f_1(u) = (1 + (f_3(u) + 1) \cdot f_2(u)) \cdot v + f_4'(u)$  &  $f_4'(u) < 1 + (f_3(u) + 1 \cdot f_2(u))$ . Define  $f_5(u)$  to be the least such  $v$ . Clearly  $f_5$  is recursive, and

$$f_1^* = {}_A(1^* + {}_A(f_3^* + {}_A 1^*) \cdot {}_A f_2^*) \cdot {}_A f_5^* + {}_A f_4^* \text{ \& } f_4^* < {}_A 1^* + {}_A(f_3^* + {}_A 1^*) \cdot {}_A f_2^*.$$

Hence  $\mathcal{R}/\bar{A} \models Bt(f_1^*, f_2^*, f_3^*, f_4^*)$ , so  $Bt$  serves  $\beta$ .

Now by [8, p. 132],  $f$  is representable by

$$\begin{aligned} S(\mathbf{x}_1, \dots, \mathbf{x}_{n+2}) : (\exists \mathbf{y}_1)(\exists \mathbf{y}_2)[&((\exists \mathbf{y}_3)(Bt(\mathbf{y}_1, \mathbf{y}_2, \mathbf{0}, \mathbf{y}_3) \text{ \& } F(\mathbf{x}_1, \dots, \mathbf{x}_n, \mathbf{y}_3))) \\ &\text{\& } Bt(\mathbf{y}_1, \mathbf{y}_2, \mathbf{x}_{n+1}, \mathbf{x}_{n+2}) \\ &\text{\& } (\mathbf{y}_4)(\mathbf{y}_4 < ' \mathbf{x}_{n+1} \Rightarrow (\exists \mathbf{y}_5)(\exists \mathbf{y}_6)(Bt(\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_4, \mathbf{y}_5) \\ &\text{\& } Bt(\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_4 + ' \mathbf{1}, \mathbf{y}_6) \text{ \& } G(\mathbf{x}_1, \dots, \mathbf{x}_n, \mathbf{y}_5, \mathbf{y}_6)))]]. \end{aligned}$$

Assume  $\mathcal{R}/\bar{A} \models S(f_1^*, \dots, f_{n+2}^*)$ . Then there are  $g_1^*$ ,  $g_2^*$ , and  $g_3^*$  such that

$$\begin{aligned} \mathcal{R}/\bar{A} \models Bt(g_1^*, g_2^*, 0^*, g_3^*) \text{ \& } F(f_1^*, \dots, f_n^*, g_3^*) \text{ \& } Bt(g_1^*, g_2^*, f_{n+1}^*, f_{n+2}^*) \\ \text{\& } (g_4^*)(g_4^* < {}_A f_{n+1}^* \Rightarrow (\exists g_5^*)(\exists g_6^*) \\ (Bt(g_1^*, g_2^*, g_4^*, g_5^*) \text{ \& } Bt(g_1^*, g_2^*, g_4^* + {}_A 1^*, g_6^*) \\ \text{\& } G(f_1^*, \dots, f_n^*, g_4^*, g_5^*, g_6^*))). \end{aligned}$$

*Subcase 1.*  $f_{n+1}^* = {}_A 0^*$ . Then since  $\mathcal{R}/\bar{A} \models F(f_1^*, \dots, f_n^*, g_3^*)$ , we must have  $F(f_1(u), \dots, f_n(u), g_3(u))$  true in  $\mathcal{N}$  for almost all  $u \in \bar{A}$ . Let us fix such a  $u \in \bar{A}$ , such that  $f_{n+1}(u) = 0$ . By (4), there is a unique  $w$  such that  $F(f_1(u), \dots, f_n(u), w)$  is true in  $\mathcal{N}$ , so  $w = g_3(u)$ . Since  $g(f_1(u), \dots, f_n(u))$  is defined, by (3) we must have  $w = g(f_1(u), \dots, f_n(u))$ . Hence  $g_3(u) = g(f_1(u), \dots, f_n(u))$  for all but finitely many  $u \in \bar{A}$ . But since  $f_{n+1}^* = {}_A 0^*$  and since  $\mathcal{R}/\bar{A} \models Bt(g_1^*, g_2^*, 0^*, g_3^*) \text{ \& } Bt(g_1^*, g_2^*, f_{n+1}^*, f_{n+2}^*)$ , and since  $Bt$  represents a function, we must have  $g_3^* = {}_A f_{n+2}^*$ , hence for all but finitely many  $u \in \bar{A}$ ,  $f_{n+2}(u) = g_3(u) = g(f_1(u), \dots, f_n(u)) = f(f_1(u), \dots, f_n(u), 0) = f(f_1(u), \dots, f_{n+1}(u))$ , so we are done in this subcase.

*Subcase 2.*  $f_{n+1}^* \neq {}_A 0^*$ . Let  $g_4(u)$  be  $f_{n+1}(u) - 1$  if  $f_{n+1}(u) \neq 0$ , and let  $g_4(u)$  be 0 otherwise. Then we can find  $g_5^*$  and  $g_6^*$  such that

$$\begin{aligned} \mathcal{R}/\bar{A} \models Bt(g_1^*, g_2^*, 0^*, g_3^*) \text{ \& } F(f_1^*, \dots, f_n^*, g_3^*) \text{ \& } Bt(g_1^*, g_2^*, f_{n+1}^*, f_{n+2}^*) \\ \text{\& } Bt(g_1^*, g_2^*, g_4^*, g_5^*) \text{ \& } Bt(g_1^*, g_2^*, g_4^* + {}_A 1^*, g_6^*) \\ \text{\& } G(f_1^*, \dots, f_n^*, g_4^*, g_5^*, g_6^*). \end{aligned}$$

Notice that  $g_4^* + {}_A 1^* = {}_A(g_4 + 1)^* = {}_A f_{n+1}^*$ . Since  $\mathcal{R}/\bar{A} \models Bt(g_1^*, g_2^*, f_{n+1}^*, f_{n+2}^*) \text{ \& } Bt(g_1^*, g_2^*, g_4^* + {}_A 1^*, g_6^*)$ , and since  $Bt$  represents a function, we must have  $g_6^* = f_{n+2}^*$ . Now since  $\mathcal{R}/\bar{A} \models G(f_1^*, f_2^*, \dots, f_n^*, g_4^*, g_5^*, g_6^*)$  and  $G$  represents  $h$ , by (3) and (4)  $g_6(u) = h(f_1(u), \dots, f_n(u), g_4(u), g_5(u))$  for all but finitely many  $u \in \bar{A}$ . Since  $Bt$  represents a function,  $g_5^*$  and  $g_6^*$  are uniquely determined by  $g_1^*$ ,  $g_2^*$ , and  $g_4^*$ . If we can show that  $g_5(u) = f(f_1(u), \dots, f_n(u), g_4(u))$  a.e. in  $\bar{A}$ , then we will have completed this subcase, since then we will have

$$\begin{aligned} f_{n+2}(u) &= g_6(u) = h(f_1(u), \dots, f_n(u), g_4(u), g_5(u)) \\ &= h(f_1(u), \dots, f_n(u), f_{n+1}(u) - 1, f(f_1(u), \dots, f_n(u), f_{n+1}(u) - 1)) \\ &= f(f_1(u), \dots, f_{n+1}(u)) \text{ a.e. in } \bar{A}. \end{aligned}$$

We now will show that  $g_5(u) = f(f_1(u), \dots, f_n(u), g_4(u))$  a.e. in  $\bar{A}$ . For assume not. For each  $u$ , let  $g'_4(u)$  be the least  $u \leq g_4(u)$  such that if  $g'_5(u) = \beta(g_1(u), g_2(u), g'_4(u))$ , then  $g'_5(u) \neq f(f_1(u), \dots, f_n(u), g'_4(u))$  if such a  $u$  exists, and  $g_4(u)$  otherwise. Let  $g'_5(u)$  be as above, and  $g'_6(u) = \beta(g_1(u), g_2(u), g'_5(u) + 1)$ . Clearly,  $g'_4$ ,  $g'_5$ , and  $g'_6$  are recursive functions. Now if  $h_4$  is such that  $h_4^* <_A g_4^*$ ,  $h_5 = \beta(g_1, g_2, h_4)$ , and  $h_6 = \beta(g_1, g_2, h_4 + 1)$ , then

$$\begin{aligned} & Bt(g_1(u), g_2(u), 0, g_3(u)) \\ & \quad \& F(f_1(u), \dots, f_n(u), g_3(u)) \& Bt(g_1(u), g_2(u), f_{n+1}(u), f_{n+2}(u)) \\ & \quad \& Bt(g_1(u), g_2(u), h_4(u), h_5(u)) \\ & \quad \& Bt(g_1(u), g_2(u), g_4(u) + 1, h_6(u)) \\ & \quad \& G(f_1(u), \dots, f_n(u), h_4(u), h_5(u), h_6(u)) \quad \text{a.e. in } \bar{A}, \end{aligned}$$

since  $h_4^* <_A g_4^* \leq_A g_4^* <_A f_{n+1}^*$ . Hence  $\mathcal{R}/\bar{A} \models S(f_1^*, \dots, f_n^*, g_4^* + 1, g_6^*)$ . But if  $g'_5(u) = f(f_1(u), \dots, f_n(u), g'_4(u) - 1)$ , and if  $g''_6(u) = \beta(g_1(u), g_2(u), g_4(u) - 1)$ , then

$$g''_6(u) = h(f_1(u), \dots, f_n(u), g'_4(u) - 1, g'_5(u)) = f(f_1(u), \dots, f_n(u), g'_4(u)).$$

But  $g''_6(u) = g'_5(u)$ , hence  $g'_5(u) = f(f_1(u), \dots, f_n(u), g'_4(u))$ . This shows that for all but finitely many  $u \in \bar{A}$ ,  $g'_4(u) = g_4(u)$ , hence  $g'_5(u) = g_5(u)$ , and  $g_5(u) = f(f_1(u), \dots, f_n(u), g_4(u))$  which is what was to be shown.

To complete this case, we must now show that if  $f(f_1(u), \dots, f_{n+1}(u)) = f_{n+2}(u)$  a.e. in  $\bar{A}$ , then  $\mathcal{R}/\bar{A} \models S(f_1^*, \dots, f_{n+2}^*)$ . Assume we are given some recursive inductive ordering of all ordered pairs of natural numbers. For each  $u$ , let  $\langle u_1(u), v_1(u) \rangle$  be the least ordered pair such that

$$\begin{aligned} w &= \beta(u_1, v_1, 0) \& g(f_1(u), \dots, f_n(u)) = w \& \beta(u_1, v_1, f_{n+1}(u)) \\ &= f_{n+2}(u) \& (x)(x < f_{n+1}(u) \Rightarrow \beta(u_1, v_1, x) \\ &= y \& \beta(u_1, v_1, x + 1) = z \& z = h(f_1(u), \dots, f_n(u), x, y)). \end{aligned}$$

Since  $f$  is representable by  $S$ , it is clear that such  $u_1$  and  $v_1$  exist if  $f(f_1(u), \dots, f_{n+1}(u)) = f_{n+2}(u)$ . Define

$$\begin{aligned} g_1(u) &= u_1(u) \quad \text{if } f(f_1(u), \dots, f_{n+1}(u)) = f_{n+2}(u), \\ &= 0 \quad \text{otherwise,} \end{aligned}$$

and

$$\begin{aligned} g_2(u) &= v_1(u) \quad \text{if } f(f_1(u), \dots, f_{n+1}(u)) = f_{n+2}(u), \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

For each recursive function  $h$  such that  $h^* <_A f_{n+1}^*$ , let  $g_3(u) = \beta(g_1(u), g_2(u), 0)$ ,  $g_5^h(u) = \beta(g_1(u), g_2(u), h(u))$ , and  $g_6^h(u) = \beta(g_1(u), g_2(u), h(u) + 1)$ . It is clear that  $g_3$ ,  $g_5^h$ , and  $g_6^h$  are recursive, and that  $\mathcal{R}/\bar{A} \models S(f_1^*, \dots, f_{n+2}^*)$ .

*Case 6.*  $f$  is the  $\mu$ -operator: Assume for all  $x_1, \dots, x_n$ , there is a  $y$  such that  $g(x_1, \dots, x_n, y) = 0$ , where  $g$  is a recursive function. Let

$$f(x_1, \dots, x_n) = \mu y [g(x_1, \dots, x_n, y) = 0].$$

Then by [8, p. 134] if  $g$  is representable and served by  $F(\mathbf{x}_1, \dots, \mathbf{x}_{n+2})$ , then  $f$  is representable by

$$S(\mathbf{x}_1, \dots, \mathbf{x}_{n+1}): F(\mathbf{x}_1, \dots, \mathbf{x}_{n+1}, 0) \ \& \ (\mathbf{y})(\mathbf{y} < \mathbf{x}_{n+1} \Rightarrow \neg F(\mathbf{x}_1, \dots, \mathbf{x}_n, \mathbf{y}, 0)).$$

Assume  $\mathcal{R}/\bar{A} \models S(f_1^*, \dots, f_{n+1}^*)$ . Then  $\mathcal{R}/\bar{A} \models F(f_1^*, \dots, f_{n+1}^*, 0^*)$ , and

$$\mathcal{R}/\bar{A} \models \neg F(f_1^*, \dots, f_n^*, g_1^*, 0)$$

if  $g_1^* <_A f_{n+1}^*$ . Clearly  $g(f_1(u), \dots, f_{n+1}(u)) = 0$  for all but finitely many  $u \in \bar{A}$ . For each  $u$ , define  $h(u)$  to be the least  $x <_{f_{n+1}(u)}$  such that  $g(f_1(u), \dots, f_n(u), x) = 0$  if such an  $x$  exists, and  $f_{n+1}(u)$  otherwise. Clearly  $h^* \leq_A f_{n+1}^*$  and  $h$  is recursive. Hence for each  $u$ ,  $h(u) = f_{n+1}(u)$  or  $g(f_1(u), \dots, f_n(u), h(u)) = 0$ . Since

$$\{u : g(f_1(u), \dots, f_n(u), h(u)) = 0\}$$

is recursive and  $\bar{A}$  is  $r$ -cohesive,

$$\mathcal{R}/\bar{A} \models \neg F(f_1^*, \dots, f_n^*, h^*, 0^*) \Leftrightarrow \neg \mathcal{R}/\bar{A} \models F(f_1^*, \dots, f_n^*, h^*, 0^*).$$

Hence we must have  $h(u) = f_{n+1}(u)$  a.e. in  $\bar{A}$ . Hence  $f_{n+1}(u) = h(u) = f(f_1(u), \dots, f_n(u))$  a.e. in  $\bar{A}$ .

Conversely, assume  $f_{n+1}(u) = f(f_1(u), \dots, f_n(u))$  a.e. in  $\bar{A}$ . Then clearly

$$\mathcal{R}/\bar{A} \models F(f_1^*, \dots, f_{n+1}^*, 0^*)$$

and for all  $h^* <_A f_{n+1}^* \neg \mathcal{R}/\bar{A} \models F(f_1^*, \dots, f_n^*, h^*, 0^*)$ . By a remark above, it follows that  $\mathcal{R}/\bar{A} \models S(f_1^*, \dots, f_{n+1}^*)$ . Hence (8) holds.

We must now show that every recursive predicate is served by some formula  $S$  of  $L$ . Let  $P$  be any  $n$ -ary recursive predicate. Then the characteristic function of  $P$ ,  $C_P$ , is a recursive function which is served and represented by some formula  $G(\mathbf{x}_1, \dots, \mathbf{x}_{n+1})$  of  $L$ . By [8, p. 120],  $P$  is expressible by

$$S(\mathbf{x}_1, \dots, \mathbf{x}_n): G(\mathbf{x}_1, \dots, \mathbf{x}_n, 0).$$

We must show that  $S$  serves  $P$ .

Assume  $\mathcal{N} \models S(k_1, \dots, k_n)$ . Then  $\mathcal{N} \models G(k_1, \dots, k_n, 0)$ . By (7),  $C_P(k_1, \dots, k_n) = 0$ , hence  $P(k_1, \dots, k_n)$  holds. Now assume  $P(k_1, \dots, k_n)$  holds. Then  $C_P(k_1, \dots, k_n) = 0$ , hence by (7),  $\mathcal{N} \models G(k_1, \dots, k_n, 0)$ , i.e.,  $\mathcal{N} \models S(k_1, \dots, k_n)$ . Thus (5) is satisfied.

Now assume recursive functions  $f_1, \dots, f_n$  and an  $r$ -cohesive set  $\bar{A}$  is given. Assume  $\mathcal{R}/\bar{A} \models S(f_1^*, \dots, f_n^*)$ . Then  $\mathcal{R}/\bar{A} \models G(f_1^*, \dots, f_n^*, 0)$ . By (8),

$$C_P(f_1(u), \dots, f_n(u)) = 0 \quad \text{a.e. in } \bar{A},$$

hence  $P(f_1(u), \dots, f_n(u))$  is true for all but finitely many  $u \in \bar{A}$ . Conversely, if  $P(f_1(u), \dots, f_n(u))$  holds for all but finitely many  $u \in \bar{A}$ , then  $C_P(f_1(u), \dots, f_n(u)) = 0$  a.e. in  $\bar{A}$ , hence by (8),  $\mathcal{R}/\bar{A} \models G(f_1^*, \dots, f_n^*, 0^*)$ , so  $\mathcal{R}/\bar{A} \models S(f_1^*, \dots, f_n^*)$ . This completes the proof of the lemma.



**THEOREM 2.1** (FEFERMAN-SCOTT-TENNENBAUM). *There is a sentence  $\Phi$  of  $L$  such that  $\mathcal{N} \models \Phi$  but  $\mathcal{R}/\bar{A} \models \neg \Phi$  for any  $r$ -cohesive set  $\bar{A}$ .*

If  $\mathcal{P}$  is any homomorphic image of the recursive functions, then unless  $\mathcal{P}$  is  $\mathcal{R}/\bar{A}$  for some  $r$ -cohesive set  $\bar{A}$  or  $\mathcal{P}$  identifies  $f$  and  $g$  if  $f(n)=g(n)$  for some fixed integer  $n$ , then  $\mathcal{P}$  has zero divisors, hence cannot be a model for arithmetic. Hence Theorem 2.1 suffices, to prove the Feferman, Scott, and Tennenbaum result.

**Proof.** Since  $\leq$  and the Kleene  $T_1$  predicate are recursive predicates, there are formulas  $U$  and  $S$  of  $L$  such that  $U$  serves, and  $S$  serves  $T_1$ , by Lemma 2.1. Let

$$\Phi \equiv (\mathbf{x}_1)(\exists \mathbf{x}_2)(\mathbf{x}_3)(\mathbf{x}_4)[(U(\mathbf{x}_3, \mathbf{x}_1) \ \& \ S(\mathbf{x}_3, \mathbf{x}_1, \mathbf{x}_4)) \Rightarrow U(\mathbf{x}_4, \mathbf{x}_2)].$$

$\Phi$  is clearly true in  $\mathcal{N}$ ; it is in fact a theorem of  $\mathcal{N}$ . Assume  $\mathcal{R}/\bar{A} \models \Phi$ , where  $\bar{A}$  is an  $r$ -cohesive set. Then for all  $f^* \in \mathcal{R}/\bar{A}$ , there is a  $k^* \in \mathcal{R}/\bar{A}$  such that for all  $g^*, h^* \in \mathcal{R}/\bar{A}$ ,  $\mathcal{R}/\bar{A} \models [(U(g^*, f^*) \ \& \ S(g^*, f^*, h^*)) \Rightarrow U(h^*, k^*)]$ . Let  $f$  be the identity function,  $I$ . Then there is a recursive function  $k$  such that for all  $g^*, h^* \in \mathcal{R}/\bar{A}$ ,

$$\mathcal{R}/\bar{A} \models [(U(g^*, I^*) \ \& \ S(g^*, I^*, h^*)) \Rightarrow U(h^*, k^*)].$$

Let  $e$  be any Gödel number for  $k$ . Letting  $g$  be the constant function  $e$ , we note that  $e^* \leq_A I^*$ . Also, since  $k$  is total, there is a recursive function  $r$  such that  $\mathcal{N} \models (\mathbf{x})S(e, \mathbf{x}, r(\mathbf{x}))$ . We choose  $h$  to be  $r$ . By Lemma 2.1,  $\mathcal{R}/\bar{A} \models U(e^*, I^*) \ \& \ S(e^*, I^*, r^*)$ . Hence we must have  $\mathcal{R}/\bar{A} \models U(r^*, k^*)$ , i.e., by Lemma 2.1,  $r(u) \leq k(u)$  for all but finitely many  $u \in \bar{A}$ . But by a well-known property of the  $T_1$  predicate,  $\{e\}(u) = k(u) < r(u)$  for all  $u$ , giving us the desired contradiction. Hence  $\mathcal{R}/\bar{A} \models \neg \Phi$ .

Our next goal is to show that, in contrast with Skolem's construction where all models obtained are elementarily equivalent, we can obtain models with different elementary theories by judicious choice of different  $r$ -cohesive sets. We will work only with co- $r$ -maximal sets, i.e.,  $r$ -cohesive sets with r.e. complements.

We call a set  $S$  *supersimple* if it is r.e., and there is no recursive function  $f(n, m)$  such that for each  $n$ ,  $\lambda m f(n, m)$  is the characteristic function of a finite set  $B_n$  such that  $B_n \cap \bar{S} \neq \emptyset$ . It is not hard to show that every  $r$ -maximal set is supersimple. In [6, Corollary 2.1], Martin shows that a Turing degree  $\mathbf{a}$  is the degree of a maximal set if and only if  $\mathbf{a}$  is r.e. and  $\mathbf{a}' = \mathbf{0}''$ . He also shows [6, Theorem 3] that an r.e. Turing degree  $\mathbf{a}$  is the degree of a supersimple set if and only if  $\mathbf{a}' = \mathbf{0}''$ . It is an easy corollary of these results to show that  $\mathbf{a}$  is the Turing degree of an  $r$ -maximal set if and only if  $\mathbf{a}$  is r.e. and  $\mathbf{a}' = \mathbf{0}''$ .

We say that a set  $A$  is *retraceable*, if there is a p.r.f.  $f$ , such that if  $a_0, a_1, \dots$  are the elements of  $A$  in order of magnitude, then  $f(a_0) = a_0$ , and  $f(a_{n+1}) = a_n$ , for all  $n > 0$ .

From now on, the letter " $I$ " will denote the identity function. For any set  $C$  and function  $f$ ,  $f|_C$  will denote the restriction of  $f$  to  $C$ .

**PROPOSITION 2.1.** *If  $f$  is a recursive function,  $A$  an  $r$ -maximal set,  $f(\bar{A}) \cap \bar{A}$  is infinite, and  $f(\bar{A}) \subset \bar{A}$  a.e., then  $f|_{\bar{A}}$  and  $I|_{\bar{A}}$  differ only finitely.*

**Proof.**  $\{x : f(x) > x\}$ ,  $\{x : f(x) = x\}$ , and  $\{x : f(x) < x\}$  are all recursive, and their union is  $N$ , hence since  $A$  is  $r$ -maximal, one of these sets must contain all but finitely many elements of  $\bar{A}$ .

Assume  $\bar{A} \subseteq \{x : f(x) > x\}$  a.e. Choose  $x \in \bar{A}$  sufficiently large so that

$$(y)((y \in \bar{A} \ \& \ y \geq x) \Rightarrow (f(y) \in \bar{A} \ \& \ f(y) > y)).$$

Define  $g(0) = x$ ,  $g(n+1) = f^{n+1}(x)$ . Then  $g$  is a recursive function which is strictly increasing, hence its range,  $B$ , is a recursive set. Also,  $B$  is infinite, and  $B \subset \bar{A}$ , contradicting the assumption that  $A$  is  $r$ -maximal.

Assume  $\bar{A} \subseteq \{x : f(x) < x\}$  a.e. We show that  $\bar{A}$  has an infinite subset which is retraced by a total recursive function, contradicting the result announced by Martin [6, p. 306, footnote 2]<sup>(2)</sup>, which implies that if a set is  $r$ -maximal, then its complement has no infinite subset which is retraced by a total recursive function.

Choose  $x \in \bar{A}$  sufficiently large so that

$$(y)((y \in \bar{A} \ \& \ y \geq x) \Rightarrow (f(y) \in \bar{A} \ \& \ f(y) < y)).$$

Let  $x_0, \dots, x_m$  be those elements of  $\bar{A}$  in order of magnitude which are less than  $x$ . Define a recursive function  $g$  by

$$\begin{aligned} g(z) &= z && \text{if } z = x_0 \vee \dots \vee z = x_m, \\ &= f(z) && \text{otherwise.} \end{aligned}$$

Note that  $g$  is a total recursive function. We show that  $g$  retraces an infinite subset of  $\bar{A}$ . Since for all  $z \in \bar{A}$ ,  $g(z) \in \bar{A}$ , and since  $(z \in \bar{A} \ \& \ z > x) \Rightarrow (g(z) < z)$ , we conclude that for every  $z \in \bar{A}$ , there exists an integer  $n$  such that  $g^n(z) \in \{x_0, \dots, x_m\} = B$ . Let  $a_0$  be the least element of  $B$  such that for infinitely many  $z \in \bar{A}$ ,  $(\exists n)(g^n(z) = a_0)$ . Assume now that we have defined  $a_i$  for all  $i \leq n$ , and that infinitely many members of  $\bar{A}$  map down to  $a_n$  under  $g$ . Look at  $\{z \in \bar{A} : g(z) = a_n\}$ . This set must be finite, for otherwise  $\{y : g(y) = a_n\}$  is recursive and would intersect  $\bar{A}$  infinitely, hence  $g(y) = a_n$  a.e. in  $\bar{A}$  and  $g(\bar{A})$  would be finite, so  $f(\bar{A})$  would be finite contradicting our assumption that  $f(\bar{A}) \cap \bar{A}$  is infinite. Hence there must be a  $z \in \bar{A}$  such that  $g(z) = a_n$ , and infinitely many members of  $\bar{A}$  map down to  $z$  under a power of  $g$ . Define  $a_{n+1}$  to be the least such  $z$ . Letting  $C = \{a_0, a_1, \dots\}$ , we note that  $C$  is an infinite retraceable subset of  $\bar{A}$ , retraced by the total recursive function  $g$ , contradiction.

Hence we conclude that there exists a finite set  $F$  such that

$$\bar{A} \subseteq \{x : f(x) = x\} \cup F,$$

and thus  $f|_{\bar{A}}$  and  $I|_{\bar{A}}$  differ only finitely. This completes the proof of the proposition.

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<sup>(2)</sup> We thank the referee for bringing this result to our attention. It states that a set  $A$  has no infinite subset retraced by a total recursive function if and only if there is no recursive function  $f(m, n)$  such that the  $\lambda n f(m, n)$  are characteristic functions of mutually disjoint sets, each of which has nonempty intersection with  $A$ . It is not hard to show that  $r$ -cohesive sets have this property.

PROPOSITION 2.2. *Let  $A_1$  and  $A_2$  be  $r$ -maximal sets. If  $f$  and  $g$  are recursive functions, nonconstant a.e. on  $\bar{A}_1$  and  $\bar{A}_2$  respectively, and  $f(\bar{A}_1) \subset \bar{A}_2$  a.e.,  $g(\bar{A}_2) \subset \bar{A}_1$  a.e., then they induce a recursive isomorphism between  $\mathcal{R}/\bar{A}_1$  and  $\mathcal{R}/\bar{A}_2$ . We can thus conclude under this hypothesis that  $\mathcal{R}/\bar{A}_1 \equiv \mathcal{R}/\bar{A}_2$ .*

**Proof.** Define  $f': \mathcal{R}/\bar{A}_2 \rightarrow \mathcal{R}/\bar{A}_1$  by  $f'(h^*) = (h \circ f)^*$ , and  $g': \mathcal{R}/\bar{A}_1 \rightarrow \mathcal{R}/\bar{A}_2$  by  $g'(h^*) = (h \circ g)^*$ . Note that by Proposition 2.1,  $f \circ g|_{\bar{A}_2} = I|_{\bar{A}_2}$  a.e. and  $g \circ f|_{\bar{A}_1} = I|_{\bar{A}_1}$  a.e. Hence  $g'(f'(h^*)) = g'(h \circ f)^* = (h \circ f \circ g)^* = h^*$  for  $h^* \in \mathcal{R}/\bar{A}_2$ , and  $f'(g'(h^*)) = f'(h \circ g)^* = (h \circ g \circ f)^* = h^*$  for  $h^* \in \mathcal{R}/\bar{A}_1$ . Thus in particular,  $g'$  is a one-one map from  $\mathcal{R}/\bar{A}_1$  onto  $\mathcal{R}/\bar{A}_2$ . It remains to show that for any  $n$ -ary  $F \in \{', +', \cdot', <'\}$  and any unary recursive functions  $f_1, \dots, f_n$ ,

$$\mathcal{R}/\bar{A}_2 \models F(g'(f_1^*), \dots, g'(f_n^*))$$

if and only if  $\mathcal{R}/\bar{A}_1 \models F(f_1^*, \dots, f_n^*)$ , and  $\mathcal{R}/\bar{A}_1 \models F(f'(f_1^*), \dots, f'(f_n^*))$  if and only if  $\mathcal{R}/\bar{A}_2 \models F(f_1^*, \dots, f_n^*)$ . The second follows from the first by symmetry.

$$\begin{aligned} \mathcal{R}/\bar{A}_2 \models F(g'(f_1^*), \dots, g'(f_n^*)) &\Leftrightarrow \mathcal{R}/\bar{A}_2 \models F((f_1 \circ g)^*, \dots, (f_n \circ g)^*) \\ &\Leftrightarrow \mathcal{N} \models F(f_1(g(u)), \dots, f_n(g(u))) \end{aligned}$$

for all but finitely many  $u \in \bar{A}_2$ . Also,

$$\mathcal{R}/\bar{A}_1 \models F(f_1^*, \dots, f_n^*) \Leftrightarrow \mathcal{N} \models F(f_1(v), \dots, f_n(v))$$

for all but finitely many  $v \in \bar{A}_1$ . These follow since the recursive predicates  $=$ ,  $+$ ,  $\cdot$ , and  $<$ , are served by  $='$ ,  $+'$ ,  $\cdot'$ , and  $<'$  respectively. Assume

$$\mathcal{N} \models F(f_1(g(u)), \dots, f_n(g(u)))$$

for all but finitely many  $u \in \bar{A}_2$ . Then  $B = \{x : \mathcal{N} \models F(f_1(x), \dots, f_n(x))\}$  is recursive and intersects  $\bar{A}_1$  infinitely, since for infinitely many  $x \in \bar{A}_1$ ,  $x = g(y)$  for some  $y \in \bar{A}_2$ , and for all but finitely many such  $y$ ,  $\mathcal{N} \models F(f_1(g(y)), \dots, f_n(g(y)))$ . Since  $\bar{A}_1$  is  $r$ -maximal,  $B$  is recursive, and  $B \cap \bar{A}_1$  is infinite, we must have  $\bar{A}_1 \subset B$  a.e. Hence  $\mathcal{N} \models F(f_1(v), \dots, f_n(v))$  for all but finitely many  $v \in \bar{A}_1$ . Conversely, assume  $\mathcal{N} \models F(f_1(v), \dots, f_n(v))$  for all but finitely many  $v \in \bar{A}_1$ . Then since  $v = g(y)$  with  $y \in \bar{A}_2$  for infinitely many  $v \in \bar{A}_1$ ,  $\mathcal{N} \models F(f_1(g(y)), \dots, f_n(g(y)))$  for infinitely many  $y \in \bar{A}_2$ . But  $C = \{x : \mathcal{N} \models F(f_1(x), \dots, f_n(x))\}$  is recursive, and intersects  $\bar{A}_2$  infinitely. Hence  $\bar{A}_2 \subset C$  a.e., so for all but finitely many  $u \in \bar{A}_2$ ,

$$\mathcal{N} \models F(f_1(g(u)), \dots, f_n(g(u))).$$

This completes the proof of the proposition.

We say that a set  $A$  is *many-one reducible* to a set  $B$ , if there is a recursive function  $f$  such that  $f(A) \subset B$  and  $f(\bar{A}) \subset \bar{B}$ .

COROLLARY 2.1. *If  $A_1$  and  $A_2$  are  $r$ -maximal sets, and  $f$  and  $g$  are recursive functions such that  $f: \bar{A}_1 \leq_m \bar{A}_2$ , and  $g: \bar{A}_2 \leq_m \bar{A}_1$ , then  $f$  and  $g$  are unique up to finite sets on  $\bar{A}_1$  and  $\bar{A}_2$  respectively.*

**Proof.** We apply Proposition 2.1 after noting that  $f$  cannot be constant a.e. on  $\bar{A}_1$ , and  $g$  cannot be constant a.e. on  $\bar{A}_2$ , so  $g \circ f(\bar{A}_1) \cap \bar{A}_1$  is infinite.

For any infinite set  $S$ , the *principal function* of  $S$  is that strictly increasing function which enumerates the elements of  $S$  in order of magnitude. We say that a function  $f$  *dominates* a function  $g$  if  $f(x) \geq g(x)$  for all but finitely many integers  $x$ . As in Martin [6, p. 295]. We call a set  $D$  *dense simple* if it is r.e., and the principal function of  $\bar{D}$  dominates every total recursive function. In [7, p. 273], Martin proves the existence of dense simple sets.

**PROPOSITION 2.3.** *Let  $\bar{A}$  be  $r$ -cohesive, and  $B$   $r$ -maximal. If there is no recursive function  $f$  such that  $f(\bar{A}) \cap \bar{B}$  is infinite, then  $\mathcal{R}/\bar{A} \not\equiv \mathcal{R}/\bar{B}$ .*

**Proof.** Let  $e$  be the Gödel number of a p.r.f. with domain  $B$ . Then  $x \in B \Leftrightarrow (\exists y)T_1(e, x, y)$  and  $x \in \bar{B} \Leftrightarrow (y) \neg T_1(e, x, y)$ . Let  $g$  be the principal function of  $\bar{D}$ , where  $D$  is any dense simple set. Since  $D$  is r.e., there is a recursive function  $k(x, y)$  such that  $(x)\lambda y k(x, y)$  is nondecreasing and  $\lim_y k(x, y) = g(x)$ . Note that  $g(x) = a \Leftrightarrow (y)(\exists t)(t \geq y \ \& \ k(x, t) = a)$ . We want a formula  $\Phi$  of  $L$  which will be interpreted in  $\mathcal{N}$  as  $(x)(\exists y)((x \in \bar{B} \ \& \ y = g(x)) \vee (x \in B \ \& \ y = 0))$ . But to apply Lemma 2.1, we need the scope of the quantifiers to be a recursive predicate. Let  $S$  serve the  $T_1$  predicate,  $U$  serve  $\leq$ , and  $V$  serve  $k(x, y) = z$ . We set

$$\Phi \equiv (\mathbf{x}_1)(\exists \mathbf{x}_2)(\exists \mathbf{x}_3)(\mathbf{x}_4)(\mathbf{x}_5)(\exists \mathbf{x}_6) \\ \times [(S(\mathbf{e}, \mathbf{x}_1, \mathbf{x}_3) \ \& \ \mathbf{x}_2 = '0) \vee (\neg S(\mathbf{e}, \mathbf{x}_1, \mathbf{x}_4) \ \& \ U(\mathbf{x}_5, \mathbf{x}_6) \ \& \ V(\mathbf{x}_1, \mathbf{x}_6, \mathbf{x}_2))].$$

We first show  $\mathcal{R}/\bar{A} \models \Phi$ . If  $u^*$  is a constant function and  $u \in B$ , then there is a constant  $w$  such that  $T_1(e, u, w)$ . Interpreting  $\mathbf{x}_2$  as  $0$ , we note that  $T_1(e, u, w) \ \& \ 0 = 0$  for all  $a \in \bar{A}$ , so by Lemma 2.1,  $\mathcal{R}/\bar{A} \models S(e^*, u^*, w^*) \ \& \ 0^* = '0^*$ , hence if  $\mathbf{x}_1$  is interpreted as a constant function  $u^*$  with  $u \in B$ , then  $\mathcal{R}/\bar{A} \models \Phi$ . If  $u^*$  is a constant function and  $u \in \bar{B}$ , then  $\neg T_1(e, u, x)$  for all  $x \in N$ , hence for every recursive function  $f$ ,  $\neg T_1(e, u, f(z))$  for all  $z \in N$ . Let  $\mathbf{x}_2$  be  $g(u)$ , and let  $v = g(u)$ . Then there is an integer  $s$  such that for all  $z \geq s$ ,  $k(u, z) = v$ . For each  $y^* \in \mathcal{R}/\bar{A}$ , define  $(t(y^*))^*$  to be  $s^*$  if  $y^* \leq_A s^*$ , and  $y^*$  otherwise. Then for any  $x^*, y^* \in \mathcal{R}/\bar{A}$ ,

$$\neg T_1(e, u, x(z)) \ \& \ y(z) \leq t(y^*)(z) \ \& \ k(u, t(y^*)(z)) = v$$

for all but finitely many  $z \in \bar{A}$ . Hence if  $u^*$  is a constant function and  $u \in \bar{B}$ , then  $\mathcal{R}/\bar{A} \models \Phi$ . Finally, if  $u^*$  is not a constant function, then since there is no recursive function  $f$  such that  $f(\bar{A}) \cap \bar{B}$  is infinite,  $\{x \in \bar{A} : u(x) \in \bar{B}\}$  is finite; hence defining  $w(z) = \mu s T_1(e, u(z), s)$ ,  $w|_{\bar{A}}$  can be extended to a recursive function. For the domain of  $w$  is an r.e. set containing  $\bar{A}$  a.e., hence there is a recursive set  $R$  containing  $\bar{A}$  a.e., and contained in the domain of  $w$ . Let  $\mathbf{x}_2$  be  $0$ , and let  $r^*$  be the equivalence class of this extension of  $w$ . Then  $T_1(e, u(z), r(z)) \ \& \ 0 = 0$  for all but finitely many  $z \in \bar{A}$ , hence by Lemma 2.1,  $\mathcal{R}/\bar{A} \models \Phi$ .

We must now show that  $\mathcal{R}/\bar{B} \models \neg \Phi$ . Assume not. Interpret  $\mathbf{x}_1$  as  $I^*$ , where  $I$  is the identity function. If the first disjunct of the scope of  $\Phi$  were to hold, by Lemma

2.1, we would have for some recursive function  $w$ ,  $T_1(e, z, w(z))$  for all but finitely many  $z \in \bar{B}$ , which is impossible by the choice of  $e$ . Hence there must be a  $v^* \in \mathcal{R}/\bar{B}$  such that for all  $x^*, y^* \in \mathcal{R}/\bar{B}$ , we can find a  $t^* \in \mathcal{R}/\bar{B}$  such that

$\mathcal{R}/\bar{B} \models \neg S(e^*, I^*, x^*) \& U(y^*, t^*) \& V(I^*, t^*, v^*)$ . Assume we have such a  $v^*$ . By the above comments, it is clear that  $\mathcal{R}/\bar{B} \models \neg S(e^*, I^*, x^*)$  for all  $x^* \in \mathcal{R}/\bar{B}$ . Hence for all  $y^* \in \mathcal{R}/\bar{B}$ , there must be a  $t^* \in \mathcal{R}/\bar{B}$  such that  $\mathcal{R}/\bar{B} \models U(y^*, t^*) \& V(I^*, t^*, v^*)$ . Since  $g$  dominates every recursive function,  $g$  dominates  $v+1$ , hence  $y(z) = \mu r[k(z, r) > v]$  can be extended to a recursive function. We choose the equivalence class of this function for our  $y^*$ . Then we must have for some recursive function  $t$ ,  $\mathcal{R}/\bar{B} \models U(y^*, t^*) \& V(I^*, t^*, v^*)$ , hence by Lemma 2.1,  $t(z) \geq y(z) \& v(z) = k(z, t(z))$  for all but finitely many  $z \in \bar{B}$ . But for all but finitely many integers  $z$ ,  $k(z, t(z)) \geq k(z, y(z)) > v(z)$ , hence by Lemma 2.1,  $\mathcal{R}/\bar{B} \models \neg V(I^*, t^*, v^*)$ , giving us the desired contradiction.

If  $A$  is r.e., then we say that an r.e. set  $B$  is a *major subset* of  $A$  if  $B \subseteq A$ ,  $A - B$  is infinite, and if  $C$  is r.e., and  $A \cup C = N$ , then  $N - (B \cup C)$  is finite.

Lachlan [4, Theorem 7] showed that every r.e. set has a major subset. Also, Lachlan [4, Theorem 8] and Robinson [11, Theorem 2] have shown that there exists an  $r$ -maximal set with no maximal superset. It is easy to show that a subset of a maximal set  $M$  is major if and only if it is  $r$ -maximal. It is also clear that a major subset of an  $r$ -maximal set is  $r$ -maximal.

**PROPOSITION 2.4.** *If  $A$  and  $B$  are  $r$ -cohesive sets and  $A \subseteq B$ , then  $\mathcal{R}/A = \mathcal{R}/B$ .*

**Proof.** We must show that for any recursive functions  $f$  and  $g$ ,  $f \sim g$  in  $\mathcal{R}/A$  if and only if  $f \sim g$  in  $\mathcal{R}/B$ . Assume  $f \sim g$  in  $\mathcal{R}/A$ . Then  $\{x : f(x) = g(x)\}$  contains  $A$  a.e., hence intersects  $B$  infinitely. Since this set is recursive, it must contain  $B$  a.e., so  $f \sim g$  in  $\mathcal{R}/B$ . Conversely, if  $f \sim g$  in  $\mathcal{R}/B$ , then  $\{x : f(x) = g(x)\}$  contains  $B$  a.e., so must contain  $A$  a.e. Thus  $f \sim g$  in  $\mathcal{R}/A$ .

By a similar argument, if  $F \in \{+, \cdot, <\}$ , we see that  $\mathcal{R}/A \models F(f_1^*, \dots, f_n^*)$  if and only if  $\mathcal{R}/B \models F(f_1^*, \dots, f_n^*)$  where  $f_1, \dots, f_n$  are any unary recursive functions. Hence  $\mathcal{R}/A = \mathcal{R}/B$ .

**COROLLARY 2.2.** *If  $A$  and  $B$  are  $r$ -cohesive sets, and  $A \cap B$  is infinite, then  $\mathcal{R}/A = \mathcal{R}/B$ .*

**Proof.**  $A \cap B$  is infinite and  $r$ -cohesive, hence by Proposition 2.4,

$$\mathcal{R}/A = \mathcal{R}/A \cap B = \mathcal{R}/B.$$

**COROLLARY 2.3.** *If  $M$  is a maximal set, and  $A$  is a major subset of  $M$ , then  $\mathcal{R}/\bar{M} = \mathcal{R}/\bar{A}$ .*

**Proof.** Since  $\bar{M} \subseteq \bar{A}$ , and by the above comments  $\bar{A}$  is  $r$ -cohesive, the result follows directly from Proposition 2.4.

**LEMMA 2.2.** *Let  $A$  and  $B$  be  $r$ -maximal sets. Then  $\bar{A} =_m \bar{B}$  if and only if there exist recursive functions  $f$  and  $g$ , nonconstant on  $\bar{A}$  and  $\bar{B}$  respectively, such that  $f(\bar{A}) \subseteq \bar{B}$  a.e., and  $g(\bar{B}) \subseteq \bar{A}$  a.e.*

**Proof.** If we have recursive functions  $h$  and  $k$  such that  $h : \bar{A} \leq_m \bar{B}$  and  $k : \bar{B} \leq_m \bar{A}$ , then  $h$  and  $k$  will serve as our  $f$  and  $g$  respectively.

Conversely, by Proposition 2.1, the set  $S = \{x : g \circ f(x) = x\}$  is a recursive set containing  $\bar{A}$  a.e. We can assume that  $\bar{A} \subseteq S$  since many-one reducibility is closed under finite differences. Note that  $f$  is one-one on  $S$  and  $g$  is one-one on  $f(S)$ , a recursive set. We can assume  $\bar{B} \subseteq f(S)$  by Proposition 2.1 and the closure of many-one reducibility under finite differences. Let  $a$  be any element of  $A$ , and let  $b$  be any element of  $B$ . Define

$$\begin{aligned} g'(x) &= g(x) & \text{if } x \in f(S), & & f'(x) &= f(x) & \text{if } x \in S, \\ &= a & \text{if } x \in \bar{f(S)}, & & &= b & \text{if } x \in \bar{S}. \end{aligned}$$

Then  $\bar{A} \leq_m \bar{B}$  via  $f'$  and  $\bar{B} \leq_m \bar{A}$  via  $g'$ .

**THEOREM 2.2.** *Let  $A$  be an  $r$ -maximal set, and  $B$  a maximal set. Then  $\mathcal{R}/\bar{A} \equiv \mathcal{R}/\bar{B}$  if and only if there is an r.e. set  $D$  such that  $\bar{D}$  is infinite,  $A \subseteq D$ , and  $\bar{D} =_m \bar{B}$ .*

**Proof.** If we have such a set  $D$ , then by Lemma 2.2, there are recursive functions  $f$  and  $g$ , nonconstant on  $\bar{D}$  and  $\bar{B}$  respectively, such that  $f(\bar{D}) \subseteq \bar{B}$  a.e.,  $g(\bar{B}) \subseteq \bar{D}$  a.e.,  $f(\bar{D}) \cap \bar{B}$  is infinite, and  $g(\bar{B}) \cap \bar{D}$  is infinite. By Proposition 2.2,  $\mathcal{R}/\bar{D} \equiv \mathcal{R}/\bar{B}$ . By Proposition 2.4,  $\mathcal{R}/\bar{A} \equiv \mathcal{R}/\bar{D}$ . Hence  $\mathcal{R}/\bar{A} \equiv \mathcal{R}/\bar{B}$ .

Conversely, assume  $\mathcal{R}/\bar{A} \equiv \mathcal{R}/\bar{B}$ . By Proposition 2.3, there is a recursive function  $f$  such that  $f(\bar{A}) \cap \bar{B}$  is infinite. Let  $D = A \cup \{x : f(x) \in B\}$ . Then  $D$  is  $r$ -maximal and  $f(\bar{D}) \subseteq \bar{B}$ . By Proposition 2.3, there is a recursive function  $g$  such that  $g(\bar{B}) \cap \bar{D}$  is infinite. Let  $B_1 = B \cup \{x : g(x) \in D\}$ . Then  $B_1$  is an r.e. superset of  $B$ , and  $\bar{B}_1$  is infinite. Since  $B$  is maximal,  $B$  and  $B_1$  can differ only finitely, hence  $g(\bar{B}) \subseteq \bar{D}$  a.e. By Lemma 2.2,  $\bar{D} =_m \bar{B}$ .

**THEOREM 2.3.** *If  $M_1$  and  $M_2$  are maximal sets, then  $\mathcal{R}/\bar{M}_1 \equiv \mathcal{R}/\bar{M}_2$  if and only if  $\bar{M}_1 =_m \bar{M}_2$ .*

**Proof.** Since, up to finite sets, the only  $r$ -maximal superset of a maximal set is the maximal set itself, this result follows immediately from Theorem 2.2.

**COROLLARY 2.4.** *If  $M_1$  and  $M_2$  are maximal sets with different Turing degrees, then  $\mathcal{R}/\bar{M}_1 \not\equiv \mathcal{R}/\bar{M}_2$ .*

**Proof.** If  $\bar{M}_1$  and  $\bar{M}_2$  have different Turing degrees, then they have different many-one degrees. Hence Theorem 2.3 gives the result.

**COROLLARY 2.5.** *Every Turing degree  $\mathbf{a}$  which is the degree of a maximal set contains a class of  $\aleph_0$  maximal sets  $M_1, M_2, \dots$  such that the elementary theories of the  $\mathcal{R}/\bar{M}_i$ ,  $i = 1, 2, \dots$  are pairwise distinct.*

**Proof.** This follows from Theorem 2.3, and the result [5, Theorem 1] that any Turing degree  $\mathbf{a}$  of a maximal set contains  $\aleph_0$  pairwise distinct many-one degrees of maximal sets.

**COROLLARY 2.6.** *If  $M_1$  and  $M_2$  are maximal sets, and  $A_i$  is a major subset of  $M_i$  for  $i = 1, 2$ , then  $\mathcal{R}/\bar{A}_1 \equiv \mathcal{R}/\bar{A}_2$  if and only if  $\bar{M}_1 =_m \bar{M}_2$ .*

**Proof.** This is an immediate result of Proposition 2.4 and Theorem 2.3.

**COROLLARY 2.7.** *There is an infinite sequence of  $r$ -maximal sets  $A_0 \supseteq A_1 \supseteq A_2 \cdots$  such that for all  $i$ ,  $A_i - A_{i+1}$  is infinite, and for all  $i$  and  $j$ ,  $\mathcal{R}/A_i = \mathcal{R}/A_j$ , where  $A_0$  is any  $r$ -maximal set.*

**Proof.** For all  $i > 0$ , let  $A_i$  be a major subset of  $A_{i-1}$ . The result follows from Proposition 2.4.

**COROLLARY 2.8.** *If  $A$  is a major subset of a maximal set  $M$ , then  $\bar{A} \neq_m \bar{M}$ .*

**Proof.** Let  $f$  be a many-one reduction of  $\bar{A}$  to  $\bar{M}$ ,  $g$  a many-one reduction of  $\bar{M}$  to  $\bar{A}$ . By Proposition 2.1, assuming  $\bar{A} =_m \bar{M}$ ,  $g \circ f|_{\bar{A}} = I|_{\bar{A}}$  a.e. But then  $g(\bar{M}) = \bar{A}$  a.e., and since  $g(\bar{M})$  is cohesive,  $\bar{A}$  would have to be cohesive, hence  $A$  would have to be maximal. But a major subset of a maximal set is not maximal, contradiction. Hence  $\bar{A} \neq_m \bar{M}$ .

**COROLLARY 2.9.** *If  $M$  is maximal, and  $A$  is  $r$ -maximal with no maximal superset, then  $\mathcal{R}/\bar{A} \not\equiv \mathcal{R}/\bar{M}$ .*

**Proof.** By Theorem 2.2, if  $\mathcal{R}/\bar{A} \equiv \mathcal{R}/\bar{M}$ , then there is an r.e. superset  $B$  of  $A$  such that  $\bar{B} =_m \bar{M}$ . But in the proof of Corollary 2.8, we noted that  $\bar{B} =_m \bar{M}$  implies  $B$  is maximal. But  $A$  has no maximal superset, yielding the desired contradiction.

**COROLLARY 2.10.** *If  $A$  is an  $r$ -maximal set with a maximal superset  $M$ , and  $B$  is an  $r$ -maximal set with no maximal superset, then  $\mathcal{R}/\bar{A} \not\equiv \mathcal{R}/\bar{B}$ .*

**Proof.** By Corollary 2.9,  $\mathcal{R}/\bar{M} \not\equiv \mathcal{R}/\bar{B}$ , and by Proposition 2.4,  $\mathcal{R}/\bar{M} = \mathcal{R}/\bar{A}$ .

In conclusion, we note that, restricting our attention to maximal sets, by Corollary 2.4 and Corollary 2.5, there are many different elementary theories of structures  $\mathcal{R}/\bar{M}$  obtained from maximal sets with the same Turing degree. However, different Turing degrees do not yield maximal sets which give structures with the same elementary theory.

We do not know if such a result holds if we consider  $r$ -maximal sets. Certainly there are many different elementary theories of structures obtained from  $r$ -maximal sets with the same Turing degree. We conjecture that different Turing degrees do in fact yield  $r$ -maximal sets which produce identical elementary theories. The following conjectures seem reasonable to us:

(I) For any maximal set  $M$ ,  $\mathbf{a}$  is the Turing degree of a major subset of  $M$  if and only if  $\mathbf{a}$  is r.e. and  $\mathbf{a}' = \mathbf{0}''$ . The only if part is clear, since  $r$ -maximal sets inhabit only such degrees.

(II) If  $A$  is any  $r$ -maximal set with no maximal superset, then  $\mathbf{a}$  is the Turing degree of a major subset of  $A$  if and only if  $\mathbf{a}$  is r.e., and  $\mathbf{a}' = \mathbf{0}''$ . Again, the only if part is clear<sup>(3)</sup>.

<sup>(3)</sup> The author has succeeded in proving (I) and (II).

(III) For any r.e. Turing degree  $\mathbf{a}$  such that  $\mathbf{a}' = \mathbf{0}''$ , there exists an infinite sequence  $A_1, A_2, \dots$  of  $r$ -maximal sets with no maximal superset all having degree  $\mathbf{a}$ , such that for all  $i$  and  $j$ ,  $i \neq j$ ,  $\mathcal{R}/\overline{A_i} \not\equiv \mathcal{R}/\overline{A_j}$ .

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